Partial Differential Equations: Final Exam

Aletta Jacobshal 03, Monday 4 April 2016, 14:00 - 17:00

Duration: 3 hours

- Solutions should be complete and clearly present your reasoning.
- 10 points are "free". There are 6 questions and the total number of points is 100. The final exam grade is the total number of points divided by 10.
- Do not forget to very clearly write your **full name** and **student number** on the envelope.
- Do not seal the envelope.

Question 1 (14 points)

Consider the equation

$$2y\, u_x + u_y = 0, \tag{1}$$

where $u = u(x, y)$.

(a) (10 points) Find the general solution of Eq. (1).

(b) (4 points) Find the solution of Eq. (1) with the auxiliary condition $u(x, 0) = \sin(x^2 + 1)$.

Question 2 (14 points)

Consider the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0. \tag{2}$$

(a) (4 points) What is the type (elliptic / hyperbolic / parabolic) of Eq. (2)? Explain your answer.

(b) (10 points) Find a linear transformation $(x, y) \to (s, t)$ that reduces Eq. (2) to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$. Express the "old" coordinates (x, y) in term of the "new" coordinates (s, t).

Question 3 (16 points)

Consider the eigenvalue problem $-X''(x) = \lambda X(x)$, $0 \leq x \leq 2\pi$, with periodic boundary conditions $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$.

(a) (6 points) Show that the given boundary conditions are of the form

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0,$$
$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0,$$

and that if a function $f(x)$ satisfies the given boundary conditions then

$$f(x)f'(x)\big|_0^{2\pi} = 0.$$

What can you conclude from these facts about the eigenvalues in this problem?

(b) (10 points) It is given that all eigenvalues are real. Prove that they are given by $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ and give the corresponding eigenfunctions.

Question 4 (14 points)

Consider the partial differential equation

$$u_{xx} + u_{yy} = -Eu, \tag{4}$$

in the domain $0 < x < a$, $0 < y < b$. Here $E > 0$ is constant.

(a) (8 points) Separate variables using a solution of the form $u(x,y) = X(x)Y(y)$ and find the ordinary differential equations satisfied by $X(x)$ and by $Y(y)$.

(b) (6 points) Assume now that the solution of Eq. (4) satisfies homogeneous Dirichlet boundary conditions, that is, $u(x,0) = u(x,b) = u(0,y) = u(a,y) = 0$. Show that Eq. (4) has a solution only if

$$E = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2},$$

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$.

It is given that the eigenvalues for the problem $-Z'' = \mu Z$ with $Z(0) = Z(c) = 0$ are $\mu_n = n^2\pi^2/c^2$, $n = 1, 2, 3, \ldots$.

Question 5 (16 points)

Consider the function $f(x) = x^3$, $x \in [0,1]$, and its Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

(a) (4 points) Check if the Fourier series converges to $f(x)$ in the L^2 sense in the interval $[0,1]$.

(b) (6 points) What is the pointwise limit of the Fourier series for $x \in [-2,2]$?

(c) (2 points) Draw the graph of the Fourier series for $x \in [-2,2]$.

(d) (4 points) At which points in $[0,2]$ does the Gibbs phenomenon appear in the Fourier series and what is the overshoot at these points?

Question 6 (16 points)

(a) (8 points) Suppose that u is a harmonic function in the disk $D = \{r < 1\}$ and that for $r = 1$ we have $u(1,\theta) = 1 + 5\sin\theta + 3\cos 2\theta$. Find the solution $u(r,\theta)$ for $r \leq 1$ and show that $u(r,\theta) \leq 9$ for $r \leq 1$.

It is given that the solution to the Laplace equation inside the disk $r < a$ has the form

$$u(r,\theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n}\left(C_n \cos(n\theta) + D_n \sin(n\theta)\right).$$

(b) (8 points) Suppose that a function w satisfies the advection-diffusion equation $w_t + 2w_x = w_{xx}$ for $0 < x < 1$ and $t > 0$ together with Robin boundary conditions $w_x = 2w$ at $x = 0$ and $x = 1$, and the initial condition $w(x,0) = 6x$, for $0 < x < 1$. Show that the *total mass*, defined by

$$M(t) = \int_0^1 w(x,t)\,dx,$$

satisfies $dM(t)/dt = 0$ and deduce that $M(t) = 3$ for all $t \geq 0$.

End of the exam (Total: 90 points)